AD 274 230

AFOSR 1741

A TECHNIQUE FOR THE SYNTHESIS OF LINEAR, NONSTATIONARY FEEDBACK SYSTEMS
PART I
THE APPROXIMATION PROBLEM

A. R. Stubberud

(Report No. 61-51)
University of California
Department of Engineering
Los Angeles, California

October 1961

CONTRACT NO. AF 49(638)-438

MECHANICS DIVISION
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
AIR RESEARCH AND DEVELOPMENT COMMAND
WASHINGTON 25, D.C.

AFOSR 1741

A TECHNIQUE FOR THE SYNTHESIS OF LINEAR, NONSTATIONARY FEEDBACK SYSTEMS

PART I

THE APPROXIMATION PROBLEM

A. R. Stubberud

(Report No. 61-51)
University of California
Department of Engineering
Los Angeles, California

October, 1961

CONTRACT NO. AF 49(638)-438

MECHANICS DIVISION
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
AIR RESEARCH AND DEVELOPMENT COMMAND
WASHINGTON 25, D.C.

FOREWORD

The research described in this report, *A Technique for the Synthesis of Linear, Nonstationary Feedback Systems--Part I. The Approximation Problem,* by A. R. Stubberud, was carried out under the technical direction of C. T. Leondes and Gerald Estrin and is part of the continuing program in Adaptive Control Systems.

This project is conducted under the sponsorship of the Air Force Office of Scientific Research of the Air Research and Development Command.

Submitted in partial fulfillment of
Contract Number AF 49(638)-438
Project Number 9783

J. M. English
Director - Engineering Research

UNIVERSITY OF CALIFORNIA
DEPARTMENT OF ENGINEERING
LOS ANGELES, CALIFORNIA

TABLE OF CONTENTS

LIST OF FIGURES

A TECHNIQUE FOR THE SYNTHESIS OF LINEAR, NONSTATIONARY FEEDBACK SYSTEMS
PART I. THE APPROXIMATION PROBLEM

A. R. Stubberud

Summary: In this paper and in a companion paper a technique for the synthesis of linear, nonstationary feedback systems is presented. This paper deals with the approximation problem, i.e., a method is developed whereby a realizable weighting function which produces a desired output when subjected to a polynomial input can be generated. Certain conditions which must be fulfilled by the outputs are also developed.

INTRODUCTION

Literally, the phrase "synthesis of a system" implies that a designer proceeds logically in a step-by-step manner from a set of system specifications to a system which will meet these specifications. In this sense many of the design techniques for control systems are not true synthesis techniques. As examples, in the field of linear, stationary control systems, the root locus and frequency domain methods are not true synthesis techniques. On the other hand, the design method proposed by Guillemin 1, in which the closed loop transfer function is first determined from the specifications and the compensation is determined subsequently is a synthesis technique.

Synthesis techniques in the areas of nonlinear control and linear, nonstationary control are few and, in general, quite specialized. No general synthesis techniques for feedback systems in either of these areas have been developed. The reason for this is quite simple. No general analytical techniques comparable to the techniques available for stationary systems are available for nonlinear and linear, nonstationary systems. The analytical techniques which are available are usually quite difficult, quite specialized or both and therefore do not lend themselves readily to application to the synthesis problem.

Engineers have put more effort into developing general approaches to the analysis of linear, nonstationary systems than they have into nonlinear systems. The two general approaches which have received the most attention are the weighting function approach 2, 3, 4 and the time varying transfer function approach. 5, 6, 7 Each of these is a logical extension of stationary system analysis. Another approach to analysis which has been discussed to a lesser degree is the differential equation approach. 8, 9, 10, 11

Synthesis techniques for nonstationary systems have tended less toward certain general approaches than have the analytical techniques. One approach which has been proposed synthesizes a given nonstationary transfer function, or a given weighting function, as an open loop system consisting of parallel elements of lower order. 12, 13

Another synthesis scheme allows a given nonstationary weighting function to be synthesized as an open loop system composed of analog computer elements.[2, 4, 14, 15] Procedures which allow a nonstationary system to be synthesized in the form of a feedback system are relatively few. Malchikov,[16], proposed one procedure by means of which a given weighting function can be synthesized as a feedback system. The procedure is applicable to the case where the system is required to contain a "fixed" plant. This method has the disadvantage that certain integral equations must be solved at specific points in the development.

One possible way of categorizing the techniques of analysis and synthesis of linear, nonstationary systems is to classify them according to the mathematical representations which are used to describe the elements of the system. In this sense the three categories would be: (1) nonstationary transfer functions; (2) weighting functions; and (3) differential equations. These categories cannot be made too rigid since in any one problem it might be desirable to use more than one representation. Each of the three categories has its own methods for combining elements of a system and each has certain advantages over the others. As was indicated in the preceding paragraph, synthesis techniques which fall into the first two categories have been developed; however none have been developed which fall into the third. The differential equation approach, however, has certain combinatorial techniques which are advantageous in the synthesis problem and the purpose of this paper is to develop a synthesis technique which exploits these combinatorial techniques.

SCOPE OF THIS PAPER AND COMPANION PAPER 18

Guillemin,[1], in 1947, proposed that the synthesis of linear, stationary feedback systems should be accomplished in three steps as follows:

(a) The closed-loop transfer function is determined from the specifications.

(b) The corresponding open-loop transfer function is found.

(c) The appropriate compensation networks are synthesized.

In this paper this synthesis technique is generalized to include linear, nonstationary systems which may, or may not, be required to contain a "fixed" plant. This technique makes use of the combinatorial techniques developed in Reference 8. The steps to be followed in the nonstationary case are:

(a) The closed loop weighting function is determined from the specifications. (It is desirable to specify the system by its weighting function since the system output for any input can then be determined by convolution.)

(b) From the closed loop weighting function determine the corresponding closed loop differential equation.

(c) Determine the open loop differential equation from the closed loop differential equation.

(d) Determine the differential equations for the appropriate compensation networks and synthesize these networks by appropriate analog computer systems.

Implementation of this procedure requires techniques for: (1) determining a weighting function which meets the given specifications; and (2) manipulating this weighting function in such a way that the appropriate compensation networks can be developed. A method is developed by means of which a weighting function can be determined which meets the specifications that: (1) the input to the system is a polynomial in time; and (2) the corresponding output approximates a given time function. In addition the application of the combinatorial manipulations developed in Reference 8 to the synthesis problem are described in detail in a related companion paper.[18] To clarify these procedures some simple examples are worked in this and the companion paper.[18]

THE APPROXIMATION PROBLEM

The first step of the synthesis problem is to determine a system weighting function which will perform according to the given specifications. This step will be called the approximation problem. Actually for the technique which is developed in this report it would be adequate to determine a system differential equation; however it is more desirable to know the system weighting function since the system output can then be determined for any input by application of the convolution integral.

There are many methods by which a system weighting function might be obtained; for instance, it might be given directly in the system specifications. In any event, once it has been established (it must have certain special properties defined below), the procedure presented in the companion paper 18 can be followed to complete the synthesis problem. In the rest of this section a specialized method for determining weighting functions is presented. The method is restricted to systems which receive inputs which are expressed as polynomials in time and whose outputs can be approximated as a separable function (see below).

This approximation method can be formulated as follows. In Figure 1 let

LINEAR SYSTEM

FIGURE 1

x(t) be a polynomial in time given by

$$x(t) = x(t-\tau) = \sum_{n=0}^{N} a_n (t-\tau)^n \qquad t \geq \tau$$
$$= 0 \qquad t < \tau \qquad (1)$$

where the a_n are constants, t is time, and τ is the time of application of x(t) to the linear system W. y(t) is the output of the linear system W which is produced as a result of the application of x(t). It is assumed that an analytic expression for y(t) can be determined from the specifications. W is the unknown linear system whose weighting function $W(t, \tau)$ is to be specified in a form which can be synthesized as a feedback system. From the convolution integral, y(t) can be expressed as

$$y(t) = \int_{-\infty}^{t} W(t,\theta)x(\theta)d\theta \tag{2}$$

Substituting Equation (1) into Equation (2), y(t) becomes

$$y(t) = y_x(t,\tau) = \sum_{n=o}^{N} a_n \int_{\tau}^{t} W(t,\theta)\Big[\theta-\tau\Big]^n d\theta; \quad t \geq \tau \tag{3}$$

$$= 0 \qquad\qquad t<\tau$$

The problem now reduces to that of solving the integral Equation (3) for $W(t,\tau)$. Fortunately this equation reduces readily to a linear, stationary differential equation which is easily solved. Prior to showing this, however, it is necessary to restrict the class of $W(t,\tau)$ which are allowed in the synthesis technique. Similar restrictions are placed on $y_x(t,\tau)$ later. It is assumed in all the succeeding works that $W(t,\tau)$ corresponds to the solution of the following linear differential equation

$$\sum_{i=o}^{m} c_i(t) \frac{\partial^i}{\partial t^i}\Big[W(t,\tau)\Big] = \sum_{j=o}^{m} b_j(t) \frac{\partial^j}{\partial t^j}\Big[\delta(t-\tau)\Big] \tag{4}$$

where $\delta(t-\tau)$ is a delta function occurring at time τ, and $c_i(t)$ and $b_j(t)$ are continuous functions of time. The properties of a $W(t,\tau)$ satisfying an equation with the form of Equation (4) have been discussed in detail previously 2, 4, 8, 10, 11, 12, 13, 14 and this discussion will not be repeated here. Suffice it to say that $W(t,\tau)$ can be written in the form

$$W(t,\tau) = W_1(t,\tau) + b_m(t)\ \delta(t-\tau) \qquad t \geq \tau \tag{5}$$

$$= 0 \qquad\qquad t<\tau$$

where $W_1(t,\tau)$ has the form

$$W_1(t,\tau) = \sum_{j=1}^{m} \beta_j(\tau)\ q_j(t) \qquad t \geq \tau \tag{6}$$

$$= 0 \qquad\qquad t<\tau$$

in which the q's must have 2n continuous derivatives and the β's must have n continuous derivatives. 8

Substituting Equation (5) into Equation (3), the relationship

$$y_x(t,\tau) = \sum_{n=o}^{N} a_n \int_{\tau}^{t} W_1(t,\tau)\Big[\theta-\tau\Big]^n d\theta + b_m(t) \sum_{n=o}^{N} a_n(t-\tau)^n \tag{7}$$

results. The solution of this equation will be broken into two parts: (1) determination of $b_m(t)$, and (2) determination of $W_1(t, \tau)$.

First, a technique for determining $b_m(t)$ will be presented. Assume that a_k $(0 \leq k \leq N)$ is that one of the non-zero a_n's with the lowest valued subscript, i.e.,

$$y_x(t,\tau) = \sum_{n=k}^{N} a_n \int_{\tau}^{t} W_1 \; (t,\theta)(\theta-\tau)^n d\theta + b_m(t) \sum_{n=k}^{N} a_n (t-\tau)^n \tag{8}$$

If now the kth partial derivative of $y_x(t,\tau)$ with respect to τ is formed, the relationship

$$\frac{\partial^k y_x(t,\tau)}{\partial\tau^k} = \sum_{n=k}^{N} a_n \int_{\tau}^{t} W_1 \; (t,\theta)(-1)^k (n)(n-1)\ldots(n+k+1)(\theta-\tau)^{n-k} d\theta +$$

$$+ \, b_m(t) a_k (-1)^k (n)(n-1)\ldots(n-k+1)(t-\tau)^{n-k} \tag{9}$$

results. Now if the limit of $\dfrac{\partial^k y_x(t,\tau)}{\partial\tau^k}$ as $\tau \to t$ is formed, the equation

$$\left.\frac{\partial^k y_x(t,\tau)}{\partial\tau^k}\right|_{\tau\to t} = b_m(t) \; a_k (-1)^k (k!) \tag{10}$$

is formed. Solving Equation (10) for $b_m(t)$, the desired relationship for $b_m(t)$ is obtained, i.e.,

$$b_m(t) = \left.\frac{(-1)^k \dfrac{\partial^k y_x(t,\tau)}{\partial\tau^k}}{(k!)a_k}\right| \; \tau \to t \tag{11}$$

Once $b_m(t)$ is known the expression

$$b_m(t) \sum_{n=o}^{N} a_n (t-\tau)^n \tag{12}$$

can then be formed. Now a new function $y_x'(t,\tau)$ is formed from Equation (8) as

$$y_x'(t,\tau) = y_x(t,\tau) - b_m(t) \sum_{n=o}^{N} a_n (t-\tau)^n = \sum_{n=o}^{N} a_n \int_\tau^t W_1(t,\theta)(\theta-\tau)^n d\theta \tag{13}$$

The second problem now is to determine $W_1(t,\tau)$ from Equation (13). First, derivatives of $y_x'(t,\tau)$ with respect to τ are formed. The first derivative is:

$$\frac{\partial y_x'(t,\tau)}{\partial\tau} = \sum_{n=o}^{N} a_n \int_\tau^t W_1(t,\theta)(-1)n(\theta-\tau)^{n-1} d\theta = a_o W_1(t,\tau) \tag{14}$$

The kth derivative ($k<N+1$) is:

$$\frac{\partial^k y_x'(t,\tau)}{\partial\tau^k} = \sum_{n=o}^{N} a_n \int_\tau^t W_1(t,\theta)(-1)^k (n)(n-1)\ldots(n-k+1)(\theta-\tau)^{n-k} d\theta$$

$$- \sum_{j=o}^{k-1} a_j (-1)^j (j!) \frac{\partial^{k-1-j} W_1(t,\tau)}{\partial\tau^{k-1-j}} \qquad k = 1, 2, \ldots, N \tag{15}$$

$$\frac{\partial^{N+1} y_x'(t,\tau)}{\partial\tau^{N+1}} = \sum_{j=o}^{N} (-1)^{j+1} (j!)\, a_j \frac{\partial^{N-j} W_1(t,\tau)}{\partial\tau^{N-j}} \tag{16}$$

Equation (16) is seen to be a linear, stationary differential equation whose solution is $W_1(t,\tau)$. To solve this differential equation uniquely N initial conditions must be obtained. Examination of Equations (15) shows that in these equations if $\tau \to t$,

$$\left.\frac{\partial^k y_x'(t,\tau)}{\partial\tau^k}\right|_{\tau\to t} = \sum_{j=o}^{k-1} (-1)^{j+1} (j!) a_j \left.\frac{\partial^{k-1-j} W_1(t,\tau)}{\partial\tau^{k-1-j}}\right|_{\tau\to t} \tag{17}$$

or

$$\left.\frac{\partial^{k-1} W(t,\tau)}{\partial\tau^{k-1}}\right|_{\tau\to t} = -\frac{1}{a_o} \left\{ \left.\frac{\partial^k y_x'(t,\tau)}{\partial\tau^k}\right|_{\tau\to t} - \sum_{j=1}^{k-2} (-1)^{j+1} (j!) a_j \left.\frac{\partial^{k-1-j} W_1(t,\tau)}{\partial\tau^{k-1-j}}\right|_{\tau\to t} \right\} \tag{18}$$

$$k = 1, 2, \ldots, N$$

Equations (18) represent a set of recursive relationships which provide the N necessary "initial conditions". Actually since the derivatives of $W_1(t, \tau)$ are with respect to τ, Equations (18) represent final conditions. This inconvenience is easily remedied by making the substitution

$$\tau = t - z \tag{19}$$

into Equations (16) and (18). The resulting equations can then be easily solved and by making the inverse substitution

$$z = t - \tau \tag{20}$$

into the solution obtained, $W_1(t, \tau)$ is recovered.

As a special case of this development Equations (11) and (16) are examined for the degenerate case of x(t), i.e.,

$$x(t-\tau) = a_N(t-\tau)^N \tag{21}$$

Equation (11) becomes

$$b_m(t) = \frac{(-1)^N \left. \dfrac{\partial y_x^N(t, \tau)}{\partial \tau^N} \right|_{\tau = t}}{(N!)\, a_N} \tag{22}$$

Equation (16) becomes:

$$\frac{\partial^{N+1} y_x'(t, \tau)}{\partial \tau^{N+1}} = (-1)^{N+1}(N!)a_N\, W_1\,(t, \tau) \tag{23}$$

or

$$W_1(t, \tau) = \frac{(-1)^{N+1} \dfrac{\partial^{N+1} y_x'(t, \tau)}{\partial \tau^{N+1}}}{(N!)a_N} \tag{24}$$

Equations (23) and (24) then should be used in the event that the input is a polynomial with but one non-zero coefficient. This case obviously includes step function inputs.

It is obvious that in order for $W(t, \tau)$ to have the form indicated in Equations (5) and (6) certain restrictions must be placed on the form of $y_x(t, \tau)$. Referring to Equation (1) let $x_n(t-\tau) = a_n(t-\tau)^n$, then $x(t-\tau)$ can be rewritten

$$x(t-\tau) = \sum_{n=o}^{N} a_n (t-\tau)^n = \sum_{n=o}^{N} x_n(t-\tau) \tag{25}$$

Equation (3) can then be rewritten in the form

$$y_x(t,\tau) = \sum_{n=o}^{N} \int_{-\infty}^{t} W(t,\theta)\, x_n(\theta)\, d\theta \tag{26}$$

However, since $W(t,\tau)$ satisfies Equation (4), $y_x(t,\tau)$ and $x(t-\tau)$ are related through the equation:

$$\sum_{i=o}^{m} c_i(t) \frac{\partial^i y_x(t,\tau)}{\partial t^i} = \sum_{j=o}^{m} b_j(t) \frac{\partial^j x(t-\tau)}{\partial t^j} = \sum_{j=o}^{m} \sum_{n=o}^{N} b_j(t) \frac{\partial^j x_n(t-\tau)}{\partial t^j} \tag{27}$$

$$= \sum_{n=o}^{N} \sum_{j=o}^{m} b_j(t) \frac{\partial^j x_n(t-\tau)}{\partial t^j}$$

Examination of the input $x(t-\tau)$ term by term reveals that each $x_n(t-\tau) = a_n(t-\tau)^n$ can be formed by integrating the delta function $(a_n)(n!)\,\delta(t-\tau)$ a total of n+1 times, i.e.,

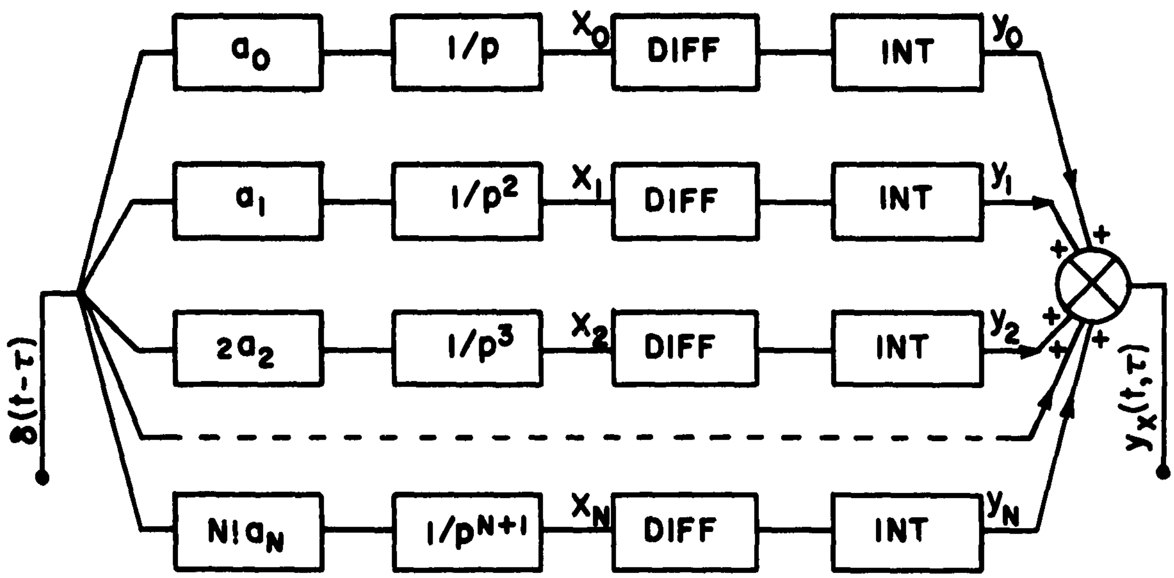
$$x_n(t-\tau) = a_n(t-\tau)^n = \int \overset{n+1}{\cdots} \int a_n\, n!\; \delta(t-\tau)\, dt \tag{28}$$

Using relationships (26), (27) and (28) a block diagram which corresponds to Equation (27) can be drawn as shown in Figure 2.

In Figure 2, y_n is that part of the output y(t) due to the $x_n(t-\tau)$ component of $x(t-\tau)$, i.e.,

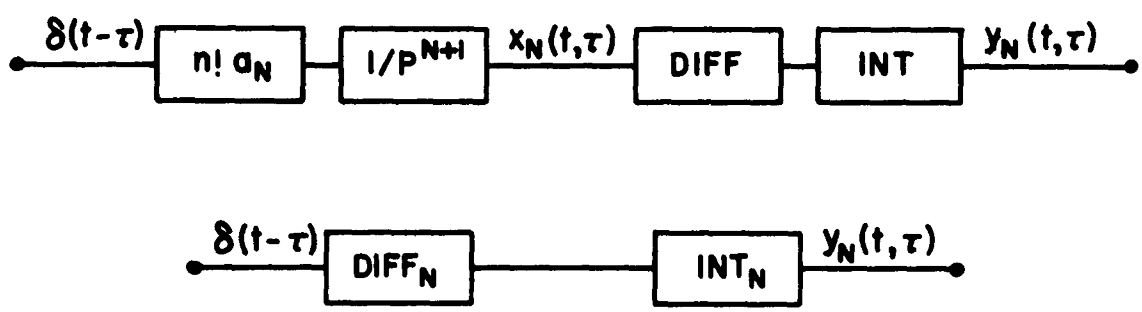
$$y_x(t,\tau) = \sum_{n=o}^{N} y_n(t,\tau) \tag{29}$$

The notation $1/p^n$ within a box in Figure 2 indicates that the input to the box is integrated n times. The system in Figure 2 apparently represents a linear system with a delta function as an input and $y_x(t,\tau)$ as an output. Figure 3(a) represents the nth branch of the system in Figure 2.



BLOCK DIAGRAM EQUIVALENT OF EQUATION (27)

FIGURE 2



n^{th} BRANCH OF FIGURE 2

FIGURE 3

Each branch then is seen to consist of a cascade combination of two linear differential equations. This combination can be replaced by a single differential equation (see the companion paper 18) as indicated in Figure 3(b). Figure 2 is then equivalent to N+1 parallel differential equations and it can be replaced by a single equivalent linear differential equation (see companion paper 18). Since the input to this system is a delta function and the output is $y_x(t,\tau)$, then $y_x(t,\tau)$ is obviously the weighting function of the equivalent linear differential equation. The first requirement on $y_x(t,\tau)$ then is that it must satisfy all of the requirements of a weighting function, i.e., it must be a separable function of t and τ with the form

$$y_x(t,\tau) = \sum_{j=o}^{M} \alpha_j(\tau)\, u_j(t) \qquad t \leq \tau$$

$$= 0 \qquad t < \tau \tag{30}$$

where the $\alpha_j(\tau)$ are linearly independent and the $u_j(t)$ are linearly independent. Obviously $y_x'(t,\tau)$ (see Equation 13) will have the same form. The possibility of $y_x(t,\tau)$ containing singularity functions (i.e., delta functions or derivatives of delta functions) is dismissed since in practice other than continuous outputs will not be encountered.

In addition to the requirement that $y_x(t,\tau)$ have the form indicated in Equation (30), examination of Equations (11), (16), and (18), which are rewritten here for convenience, reveal some additional requirements on $y_x(t,\tau)$.

$$b_n(t) = \frac{(-1)^k \left. \dfrac{\partial^k y_x(t,\tau)}{\partial \tau^k} \right|_{\tau \to t}}{(k!)a_k} \tag{11}$$

$$\frac{\partial^{N+1} y_x'(t,\tau)}{\partial \tau^{N+1}} = \sum_{j=o}^{N} (-1)^{j+1}(j!)\, a_j \frac{\partial^{N-j} W_1(t,\tau)}{\partial \tau^{N-j}} \tag{16}$$

$$\left. \frac{\partial^{k-1} W_1(t,\tau)}{\partial \tau^{k-1}} \right|_{\tau \to t} = -\frac{1}{a_o} \left\{ \left. \frac{\partial^k y_x'(t,\tau)}{\partial \tau^k} \right|_{\tau \to t} - \sum_{j=1}^{k-1} (-1)^{j+1}(j!)\, a_j \left. \frac{\partial^{k-1-j} W_1(t,\tau)}{\partial \tau^{k-1-j}} \right|_{\tau \to t} \right\} \tag{18}$$

$$k = 1, 2, \ldots, N$$

The solution for Equation (16) has the form

$$W_1(t,\tau) = \sum_{i=1}^{N} c_i e^{-\gamma_i(t-\tau)} + \int_t^{\tau} G(\sigma-\tau)(-1)^N \frac{\partial^{N+1} y_x'(t,\sigma)}{\partial\sigma^{N+1}} d\sigma \tag{31}$$

where the c_i are functions of the "initial conditions" in Equation (18); the δ's are roots of the polynomial

$$\sum_{j=o}^{N} (j!)\, a_j \gamma^{N-j} = 0 \tag{32}$$

and $G(x-\theta)$ is the solution of the differential equation

$$\sum_{j=o}^{N} (j!)\, a_j \frac{\partial^{N-j} G(x-\theta)}{\partial x^{N-j}} = \delta(x-\theta) \tag{33}$$

From Equation (30)

$$y_x'(t,\sigma) = \sum_{i=1}^{M} \alpha_j(\sigma)\, u_j(t) \tag{34}$$

and

$$\frac{\partial^{N+1} y_x'(t,\tau)}{\partial\sigma^{N+1}} = \sum_{i=1}^{M} \alpha_i^{(N+1)}(\sigma)\, u_i(t) \tag{35}$$

If $G(\sigma-\tau)$ is written in its general exponential form as

$$G(\sigma-\tau) = \sum_{r=1}^{N} T_r\, e^{-\gamma_i(\sigma-\tau)} \tag{36}$$

then $W_1(t,\tau)$ can be written in the form

$$W_1(t,\tau) = \sum_{i=1}^{N} c_i(t)\, e^{-\gamma_i(t-\tau)} + \sum_{i=1}^{N} T_i e^{\gamma_i \tau} \int_t^{\tau} e^{-\gamma_i t}(-1) \sum_{j=1}^{M} \alpha_j^{(N+1)}(\sigma) u_j(t)\, d\sigma \tag{37}$$

Rearranging Equation (37)

$$W_1(t,\tau) = \sum_{i=1}^{N} e^{\gamma_i \tau}\left[c_i(t) e^{-\gamma_i t} + (-1)^{N+1} \quad T_i \sum_{j=1}^{M} u_j(t)\, F_{ij}(t)\right]$$
$$+ \sum_{j=1}^{M} u_j(t)\left[(-1)^N \sum_{i=1}^{N} T_i e^{\gamma_i \tau} F_{ij}(\tau)\right] \tag{38}$$

where

$$F_{ij}(t) = \int^{t} e^{-\gamma_i \sigma} \alpha_j(\sigma)\, d\sigma \qquad (39)$$

Equation (38) indicates that $W_1(t,\tau)$ is a separable function of t and τ with a maximum of M+N terms; therefore it can be written

$$W_1(t,\tau) = \sum_{j=1}^{M+N} g_j(\tau)\, u_j(t) \qquad (40)$$

where the g's and u's are defined according to Equation (38). In order for $W_1(t,\tau)$ to be a weighting function of the type considered in Equation (6) it is sufficient that the g's have M+N continuous derivatives and the u's have 2(M+N) continuous derivatives. Applying these differentiability requirements to Equation (38) it can be seen that the α's should have 2M+3N continuous derivatives and the u's should have 2M+2N continuous derivatives.

Example

As an example of the technique just developed, consider the following problem. The input to a linear system has the form:

$$x(t-\tau) = 2u(t-\tau) + (t-\tau) - 2(t-\tau)^2 \qquad (41)$$

From the system specifications, the output of the system is required to have the form:

$$y_x(t,\tau) = -\frac{7}{6}t^5 + t^4\left[\frac{5}{6} + \frac{10}{3}\tau\right] + t^3\left[1 - \frac{3}{2}\tau - \tau^2\right]$$
$$+ t^2\left[2\tau + \frac{1}{2}\tau^2 + \frac{2}{3}\tau^3 + 1\right] + t\left[-3\tau^2 + \frac{1}{6}\tau^3 + \frac{1}{6}\tau^4 - \tau + 2\right] \qquad (42)$$

From the given input $x(t-\tau)$ and the desired output the weighting function $W(t,\tau)$, which represents the desired linear system, is to be determined. It was shown that $W(t,\tau)$ can, in general, be divided into two parts, i.e.,

$$W(t,\tau) = W_1(t,\tau) + b_m(t)\, \delta(t-\tau) \qquad (43)$$

The output $y_x(t,\tau)$ can then be represented by

$$y_x(t,\tau) = \sum_{n=o}^{N} a_n \int_{\tau}^{t} W_1(t,\theta)(\theta-\tau)^n d\theta + b_m(t) \sum_{n=o}^{N} a_n (t-\tau)^n \qquad (44)$$

where $N = 2$, $a_o = 2$, $a_1 = 1$, and $a_2 = -2$.

The first step in determining $W(t, \tau)$ is to find $b_m(t)$. Taking the limit of $y_x(t, \tau)$ as $t \to \tau^+$, the relationship in Equation (45) is obtained.

$$\lim_{t \to \tau} y_x(t, \tau) = 2\tau = b_m(\tau)\, a_o; \tag{45}$$

therefore

$$b_m(t) = \frac{2t}{a_o} = t \tag{46}$$

The second term of $y_x(t, \tau)$ in Equation (44) is then

$$b_m(t) \sum_{n=o}^{N} a_n (t-\tau)^n = t\left[2-\tau-2\tau^2\right] + t^2\left[1+4\tau\right] + t^3\left[-2\right] \tag{47}$$

Inserting (47) into (44) and solving for

$$y_x'(t, \tau) = y_x(t, \tau) - b_m(t) \sum_{n=o}^{N} a_n (t-\tau)^n \tag{48}$$

Equation (49) is obtained:

$$y_x'(t, \tau) = -\frac{7}{6}t^5 + t^4\left[\frac{5}{6} + \frac{10}{3}\tau\right] + t^3\left[3 - \frac{3}{2}\tau - \tau^2\right]$$

$$+ t^2\left[-2\tau + \frac{1}{2}\tau^2 + \frac{2}{3}\tau^3\right] + t\left[-\tau^2 + \frac{1}{6}\tau^3 + \frac{1}{6}\tau^4\right] \tag{49}$$

Next the derivatives of $y_x'(t, \tau)$ with respect to τ are formed

$$\frac{\partial y_x'}{\partial \tau} = \frac{10}{3}t^4 + t^3\left[-\frac{3}{2} - 2\tau\right] + t^2\left[-2 + \tau + 2\tau^2\right] + t\left[-2\tau + \frac{1}{2}\tau^2 + \frac{2}{3}\tau^3\right] \tag{50}$$

$$\frac{\partial^2 y_x'}{\partial \tau^2} = -2t^3 + t^2\left[1 + 4\tau\right] + t\left[-2 + \tau + 2\tau^2\right] \tag{51}$$

$$\frac{\partial^3 y_x'}{\partial \tau^3} = 4t^2 + t + 4t\tau \tag{52}$$

Referring to Equations (16) and (18) in the previous section it can be seen that

$$\frac{\partial^3 y_x'(t,\tau)}{\partial\tau^3} = 4t^2 + t + 4t\tau = -2a_2W_1(t,\tau) + a_1\frac{\partial W_1(t,\tau)}{\partial\tau} - a_o\frac{\partial^2 W_1(t,\tau)}{\partial\tau^2} \tag{53}$$

$$W_1(t,t^-) = -\frac{1}{a_o}\left\{\left.\frac{\partial y_x'(t,\tau)}{\partial\tau}\right|_{\tau\to t^-}\right\} = 2t^2 \tag{54}$$

$$\left.\frac{\partial W_1(t,\tau)}{\partial\tau}\right|_{\tau\to t} = -\frac{1}{a_o}\left\{\left.\frac{\partial^2 y_x'}{\partial\tau^2}\right|_{\tau\to t^-} - a_1W_1(t,t^-)\right\} = t \tag{55}$$

The solution of the set of Equations (53), (54) and (55) then is the desired weighting function. To simplify the algebra involved and to make the solution of this set of equations easier, the following substitutions are made

$$z = t - \tau$$

$$k_2 = 2t$$

$$k_1 = 4t^2 + \frac{t}{2}$$

$$f(z) = W_1(t,\ t-z) \tag{56}$$

Then since $a_o = 2$, $a_1 = 1$ and $a_2 = -2$, Equations (53), (54) and (55) can be rewritten as:

$$\frac{d^2f}{dz^2} + \frac{1}{2}\frac{df}{dz} - 2f = k_2z - k_1 \tag{53'}$$

$$f(0) = 2t^2 \tag{54'}$$

$$f'(0) = -t \tag{55'}$$

The complementary solution of Equation (53') has the form

$$f_c(x) = Ae^{-\gamma_1 x} + Be^{-\gamma_2 x} \tag{57}$$

where A and B are functions of the initial conditions and γ_1 and γ_2 are roots of the polynomial

$$\gamma^2 + \frac{1}{2}\gamma - 2 = 0 \tag{58}$$

The particular solution has the form

$$f_p = \frac{k_1}{2} - \frac{k_2}{8} - \frac{k_2}{2} \quad x \tag{59}$$

The entire solution then has the form

$$f(x) = Ae^{-\gamma_1 x} + Be^{-\gamma_2 x} + \frac{k_1}{2} - \frac{k_2}{8} - \frac{k_2}{2} \quad x \tag{60}$$

Applying the initial conditions of (54') and (55'), the constants A and B can be determined to be

$$A = \frac{\left[f(0) + \frac{k_2}{8} - \frac{k_1}{2}\right]\left[-\gamma_2\right] - f'(0) - \frac{k_2}{2}}{1 - 2} \tag{61}$$

$$B = \frac{f'(0) + \frac{k_2}{2} + \gamma_1\left[f(0) + \frac{k_2}{8} - \frac{k_1}{2}\right]}{\gamma_1 \quad -\gamma_2} \tag{62}$$

Substituting the values in (54'), (55') and (56) it can be shown that

$$A = 0$$
$$B = 0 \tag{63}$$

f(x) therefore equals the particular solution of the differential equation, i.e.,

$$f(x) = \frac{k_1}{2} - \frac{k_2}{8} - \frac{k_2}{2} x \tag{64}$$

Substituting the values of Equations (56) into (64) it is seen that

$$W_1(t,\tau) = t^2 + t\tau \tag{65}$$

The overall weighting function defined by Equation (43) is then

$$W(t,\tau) = t^2 + t\tau + t\delta(t-\tau) \tag{66}$$

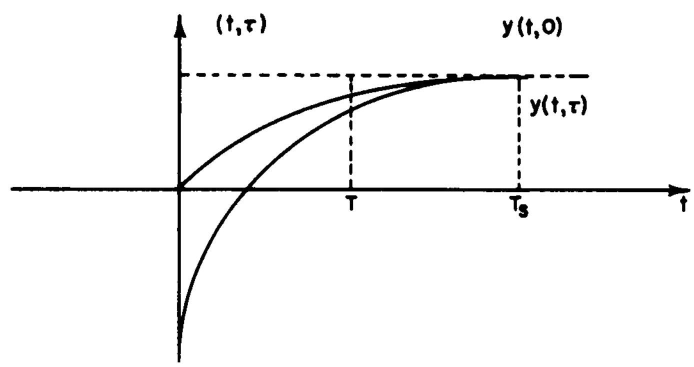
and the problem is solved. Convolution of (66) and (41) will indeed produce $y_x(t,\tau)$ as given in Equation (42).

APPROXIMATION OF SEPARABLE FUNCTIONS

From the discussion in the previous section it is obvious that for a large class of synthesis problems the response function must have the form of a separable function which satisfies suitable differentiability conditions. In addition, the class of weighting functions considered in this paper are separable functions. It is then necessary to the synthesis technique to have a method of approximating a function of two variables as a separable function of the two variables. Some work has been done on this problem; for instance, Cruz, 17, proposed an impulse-train approximation of weighting functions; and Cruz and Van Valkenberg, 13, proposed a method of approximating weighting functions as separable functions which approximate the weighting functions in some sense. The rest of this section describes a simple method of approximating a function of two variables as a separable function.

As a tool to help explain this approximation method, a final value controller will be considered. Suppose that a system is to be designed according to the following specifications. The system will receive a step input at some time (which is unknown) between $t = 0$ and $t = T$ and it is required that the system settle to within n% of its final steady state value (unity) by time T_s where $T_s > T$. On the other hand, it is desirable to keep the system as sluggish as possible at all times. Thus if the system is designed as a nonstationary linear system, the first step of the design is to determine a step response $y(t,\tau)$ of this system which satisfies the specifications. From this $y(t,\tau)$, the appropriate weighting function can be determined and the design completed.

Figure 4 represents a two-dimensional view of $y(t,\tau)$ in which the τ axis is perpendicular to the plane of the paper. The planes $y(t, 0)$ and $y(t, T)$ are the two planes which bound the region of interest of $y(t,\tau)$, i.e., if the function $y(t,\tau)$ is suitable in this



TWO DIMENSIONAL VIEW OF y(t, T)

FIGURE 4

region the specifications can be met. Suppose that $y(t, 0)$ and $y(t, T)$ are chosen with the form:

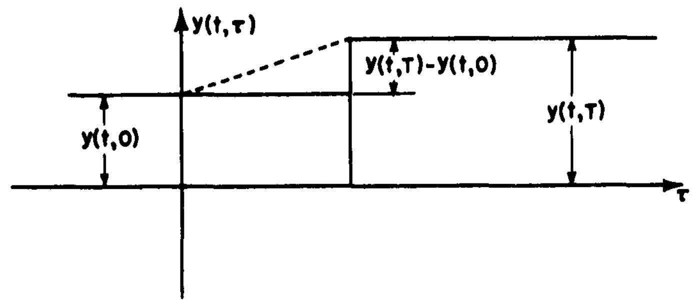
$$y(t, 0) = 1 - e^{-\alpha_1 t} \qquad y(t, T) = 1 - e^{-\alpha_2 (t-T)} \tag{67}$$

where α_1 and α_2 are chosen so that

$$y(T_s, 0) = y(t_s, T) = 1 - \frac{n}{100} \tag{68}$$

It is seen then that $y(t,\tau)$ meets the specifications for two values of τ, i.e., $\tau = 0$ and $\tau = T$. In addition $y(t,\tau)$ for $0 \leq \tau \leq T$ might be fixed for several additional values of τ; however for the purpose of this exposition two will suffice. Obviously if $y(t,\tau)$ could be represented by relationships similar to Equations (67) for all $0 \leq \tau \leq T$, the specifications could be met exactly. The number of values of τ for which $y(t,\tau)$ is fixed, for instance by Equations (67), will also be the number of terms in the final separable function; thus the better the desired approximation the higher the order of the resultant system.

Suppose Figure 4 is redrawn as in Figure 5 in which the t axis is now normal to the plane of the paper. Outside the region $0 \leq \tau \leq T$, $y(t, \tau)$ is arbitrary. At the boundaries of the region $y(t, \tau)$ satisfies Equations (67).



APPROXIMATION OF y(t, T)

FIGURE 5

If the only constrai ts on $y(t, \tau)$ are at $\tau=0$ and $\tau=T$, $y(t, \tau)$ can be approximated in several ways. For instance let

$$y(t, \tau) = y(t, 0) + \left[y(t, T) - y(t, 0)\right] u(\tau - T) \tag{69}$$

where $u(\tau - T)$ is a step function occurring at $\tau = T$. $y(t, \tau)$ then corresponds to the full line function in Figure 5. If Equation (69) is Laplace transformed with respect to τ, the resulting transform $Y(t, s)$ is given by:

$$Y(t, s) = \frac{y(t, 0)}{s} + \frac{y(t, T) - y(t, 0)}{s} e^{-sT} \tag{70}$$

Now if $\frac{e^{-sT}}{s}$ is approximated by a Pade approximation 1, for instance,

$$\frac{e^{-sT}}{s} \cong \frac{2}{T} \quad \frac{(s - 3/T)}{s\left[s^2 + 4/T^s + 6/T^2\right]} \tag{71}$$

then

$$Y(t, s) \cong Y^*(t, s) = \frac{y(t, 0)}{s} + \left[y(t, T) - y(t, 0)\right] \left\{ - \frac{2}{T} \quad \frac{(s - 3/T)}{s(s^2 + 4/T^s + 6/T^2)} \right\} \tag{72}$$

The inverse transform of $Y^*(t, s)$ will be denoted $y^*(t, \tau)$. Other Pade approximations could be used in Equation (70). Obviously a better approximation would be obtained if a higher order polynomial were used. This particular approximation was chosen because it is relatively simple and because the numerator is of lesser degree than the denominator. This latter condition forces the second term of $y^*(t, \tau)$ to be zero at $\tau = 0$, and, hence, $y^*(t, 0) = y(t, 0)$. Taking the inverse transform of $Y^*(t, s,)$, the approximation $y^*(t, \tau)$ is obtained as:

$$y^*(t, \tau) = y(t, 0)\left[e^{-2\tau/T} \sin\left(\frac{\sqrt{2}\,\tau}{T} + \psi\right)\right]$$

$$+ y(t, T)\left[1 - 3e^{-2\tau/T} \sin\left(\frac{\sqrt{2}\,\tau}{T} + \psi\right)\right] \quad t \geq \tau$$

where $\psi = \sin^{-1}(1/3)$ and $y(t, 0)$ and $y(t, T)$ are given in Equation (67).

Obviously this type of an approximation meets the differentiability requirements discussed in the previous section. As a very rough check on the accuracy of the approximation $y^*(t,\tau)$ it is seen that

$$
\begin{aligned}
y^*(t, 0) &= y(t, 0) \\
y^*(t, T) &= 0.398y(t, 0) + 0.602y(t, T) \qquad (74) \\
y^*(t, \infty) &= y(t, T)
\end{aligned}
$$

The accuracy of this type of approximation can be improved by: (1) approximating $y(t, \tau)$ by several steps in the τ direction (see Figure 5), i.e., let

$$y(t, \tau) = y(t, 0) + \sum_{n=1}^{N} \left[y(t, \frac{n}{N} T) - y(t, \left(\frac{n-1}{N}\right)) T \right] u(\tau - \frac{n}{N} T) \qquad (75)$$

where N is an integer; and (2) using a higher order Pade approximation for $e^{\frac{-n}{N} sT}$

Another method of approximation which might be used is to approximate a higher order derivative of $y(t, \tau)$ as a series of step functions in the τ direction and then integrating this function an appropriate number of times to regain $y(t, \tau)$. As an example, the derivative of the dotted line representation of $y(t, \tau)$ in Figure 5 is

$$y^{(0,1)}(t, \tau) = \frac{\partial}{\partial \tau} \left[y(t, \tau) \right] = \frac{y(t, T) - y(t, 0)}{T} \left[u(\tau) - u(t - \tau) \right] \qquad (76)$$

If this equation is transformed, the Pade approximation in Equation (7) is used, and an inverse transform is then taken; the approximation of $y^{(0,1)}(t, \tau)$ which is denoted $y^{*(0,1)}(t, \tau)$ is given by

$$y^{*(0,1)}(t, \tau) = \frac{y(t, T) - y(t, 0)}{T} \cdot 3e^{-2\tau/T} \sin\left[\frac{\sqrt{2}\,\tau}{T} + \psi\right] \qquad (77)$$

where again $\psi = \sin^{-1}(\frac{1}{3})$. Then since

$$y^*(t, \tau) = y(t, 0) + \int_0^{\tau} y^{*(01)}(t, \theta)\, d\theta \qquad (78)$$

the final approximation is:

$$
\begin{aligned}
y^*(t, \tau) &= y(t, 0)^{-2\tau/T} \left\{ \frac{1}{2} \sin\frac{\sqrt{2}\,\tau}{T} + \cos\frac{\sqrt{2}\,\tau}{T} \right\} + y(t, T) \left\{ 1 - e^{-2\tau/T} \left[\frac{1}{2} \sin\frac{\sqrt{2}\,\tau}{T} \right.\right. \\
&\qquad \left.\left. + \cos\frac{\sqrt{2}\,\tau}{T} \right] \right\} ; \quad t \geq \tau \qquad (79) \\
y^*(t, \tau) &= 0 \qquad ; \quad t < \tau
\end{aligned}
$$

As a comparison with the approximation of Equation (73), it is seen that

$$
\begin{aligned}
y^*(t,0) &= y(t,0) \\
y^*(t,T) &= 0.116y(t,0) + 0.884\, y(t,T) \\
y^*(t,\infty) &= y(t,T)
\end{aligned}
\tag{80}
$$

This procedure then produces a more accurate approximation to $y(t,\tau)$ than does the first procedure. Again a better approximation can be found if $\frac{\partial}{\partial\tau}\left[y(t,\tau)\right]$ is approximated by more terms as in the case for $y(t,\tau)$ in the first procedure (see Equation 75).

These approximating procedures, while leaving something to be desired from the standpoint of mathematical sophistication, are straightforward and easily applied to not only weighting functions and step responses but also polynomial responses which are also separable functions. The advantage of this method is that the functions obtained always have enough differentiability to apply the methods developed in the preceding section.

CONCLUSIONS

In this paper the approximation portion of a synthesis technique for linear, nonstationary feedback systems has been presented. A method for determining a weighting function which transforms a polynomial input into an output which can be described by a separable function of two variables is presented. In addition, a method is described by means of which a function of two variables can be approximated by a separable function of the two variables.

In a companion paper 18 the synthesis technique is completed in that it is shown how a weighting function can be synthesized as a feedback system.

REFERENCES

1. Truxal, J.G., Control Systems Synthesis, McGraw-Hill Book Co., New York, 1955, Chapter 5.

2. Borskii, V., "On the Properties of Impulsive Response Function of Systems with Variable Parameters", Automation and Remote Control, Vol. 20, July, 1959, pp. 822-830.

3. Zadeh, L.A., "The Determination of the Impulsive Response of Variable Networks", Journal of Applied Physics, Vol. 21, July, 1950, pp. 642-645.

4. Control Systems Engineering, Edited by William W. Siefert and Carl W. Steeg, Jr., McGraw-Hill Book Company, New York, 1960, Chapters 12 and 15.

5. Zadeh, L.A., "Circuit Analysis of Linear Varying-Parameter Networks", Journal of Applied Physics, Vol. 21, November, 1950, pp. 1171-1177.

6. Zadeh, L.A., "A General Theory of Linear Signal Transmission Systems", Journal of the Franklin Institute, April, 1952, pp. 293-312.

7. Zadeh, L.A., "Frequency Analysis of Variable Networks", Proceedings of the I.R.E., Vol. 38, March, 1950, pp. 291-299.

8. Stear, E.B. and Stubberud, A.R., "Nonstationary Signal Flow Graph Theory", Report No. 60-64, University of California, Department of Engineering, Los Angeles, California, November, 1960.

9. Darlington, S., "Nonstationary Smoothing and Prediction Using Network Theory Concepts", I.R.E. Transactions on Information Theory, Vol. IT-5, May, 1959, pp. 1-11.

10. Darlington, S., "An Introduction to Time-Variable Networks", Proceedings of Symposium on Circuit Analysis, University of Illinois, 1955, pp. 5-1 to 5-25.

11. Darlington, S., "Time-Variable Transducers", Proceedings of the Symposium on Active Networks and Feedback Systems, Polytechnic Press of the Polytechnic Institute of Brooklyn, New York, 1960, pp. 621-633.

12. Cruz, J.B., "On the Realizability of Linear Differential Systems", I.R.E. Transaction on Circuit Theory, Vol. CT-7, Sept. 1960, pp. 347-348.

13. Cruz, J.B. and Van Valkenberg, M.E., "The Synthesis of Models for Time-Varying Linear Systems", Proceedings of the Symposium on Active Networks and Feedback Systems, Polytechnic Press of the Polytechnic Institute of Brooklyn, New York, 1960, pp. 527-544.

14. Batkov, A.M., "The Problem of Synthesizing Linear Dynamic Systems with Variable Parameters", Automation and Remote Control, Vol. 19, January, 1958.

15. Matyash, I., "Methods of Analog Computer Solution of Linear Differential Equations with Variable Coefficients", Automation and Remote Control, Vol. 20, July, 1959, pp. 813-821.

16. Malchikov, S.V., "On the Synthesis of Linear Automatic Control Systems with Variable Parameters", Automation and Remote Control, Vol. 20, December, 1959, pp. 1543-1549.

17. Cruz, J.B., "A Generalization of the Impulse Train Approximation for Time-Varying Linear System Synthesis in the Time Domain", I.R.E. Transactions on Circuit Theory, Vol. CT-6, December, 1959, pp. 393-394.

18. Stubberud, A.R., "A Technique for the Synthesis of Linear Nonstationary Feedback Systems, Part II, The Synthesis Problem", to be published.

19. Stubberud, A.R., E.D. Stear, "Signal Flow Graph Theory for Linear Time Variable Systems", accepted for publication in AIEE Transactions.

DISTRIBUTION LIST

No. Copies	Agency
10	ASTIA ATTN: TIPCR Arlington Hall Station Arlington 12, Virginia
4	AFOSR ATTN: SRHM (2) ATTN: SRGL (2) Washington 25, D.C.
2	EOOAR (2) Shell Building 47 Rue Cantersteen Brussels, Belgium
1	AEDC ATTN: AEOIM Arnold Air Force Station Tennessee
1	AFFTC ATTN: FTOTL Edwards AF Base California
1	AFMDC ATTN: HDOI Holloman AF Base New Mexico
1	AFMTC ATTN: AFMTC Tech Library-MU-135 Patrick AF Base Florida
1	AFOSR (SRLTL) Holloman AF Base New Mexico
4	Aeronautical Systems Division ATTN: WWAD (2) ATTN: WWRMCM (1) ATTN: ASRNGC (1) Wright-Patterson AF Base, Ohio
2	ARL (OAR) Wright-Patterson AF Base Ohio
1	Institute of Technology Library (AU) MCLI-LIB, Bldg. 125, Area B Wright-Patterson Air Force Base, Ohio
1	APGC (PGAPI) Eglin, AFB, Florida
2	AFCRL ATTN: CRRELA L.G. Hanscomb Field Bedford, Massachusetts
1	AFSWC (SWOI) Kirtland AF Base New Mexico
1	Director BRL ATTN: Library Aberdeen Proving Ground, Maryland
1	Office of Ordnance Research Box CM Duke Station Durham, North Carolina
1	Army Rocket and Guided Missile Agency ATTN: Technical Library Redstone Arsenal, Alabama
1	Signal Corps Engineering Laboratory ATTN: SIGFMIEL - RPO Fort Monmouth, New Jersey
1	Office of the Chief of R&D ATTN: Scientific Information Department of the Army Washington 25, D.C.
4	Office of Naval Research (2) ATTN: Mechanics Branch (1) Airbranch (1) Washington 25, D.C.
1	Naval Research Laboratory ATTN: Documents Library Washington 25, D.C.
1	Naval Ordnance Laboratory ATTN: Library White Oak, Maryland
1	David Taylor Model Basin Aerodynamics Laboratory ATTN: Library Washington 7, D.C.
1	Chief, Bureau of Ordnance Department of the Navy ATTN: Special Projects Office, SP-2722 Washington 25, D.C.
1	NASA ATTN: Document Library 1502 H Street, N.W. Washington 25, D.C.
1	Ames Research Center (NASA) ATTN: Technical Library Moffett Field, California
2	AFOSR ATTN: SRHM ATTN: SRGL Washington 25, D.C.

DISTRIBUTION LIST

No. Copies	Agency
1	High Speed Flight Station (NASA) ATTN: Technical Library Edwards AFB, California
1	Langley Research Center (NASA) ATTN: Technical Library Langley AFB, Virginia
1	Lewis Research Center (NASA) ATTN: Technical Library 21000 Brookpark Road Cleveland 35, Ohio
1	National Bureau of Standards U.S. Department of Commerce ATTN: Technical Reports Section Washington 25, D.C.
1	Office of Technical Services U.S. Department of Commerce ATTN: Technical Reports Section Washington 25, D.C.
1	National Science Foundation ATTN: Engineering Sciences Division 1951 Constitution Avenue, NW Washington 25, D.C.
1	U.S. Atomic Energy Commission Technical Information Extension P.O. Box 62 Oak Ridge, Tennessee
1	U.S. Atomic Energy Commission Technical Information Service 1901 Constitution Avenue, NW Washington 25, D.C.
1	Lewis Research Center (NASA) ATTN: Technical Library 21000 Brookpark Road Cleveland 35, Ohio
2	Southwest Research Institute ATTN: Applied Mechanics Reviews 8500 Culebra Road San Antonio 6, Texas
1	Aeronautical Engineering Review 2 East 64th Street New York 21, New York
1	Institute of Aeronautical Sciences ATTN: Library 2 East 64th Street New York 21, New York
1	Chairman Canadian Joint Staff (DRB/DSIS) 2450 Massachusetts Avenue, NW Washington 25, D.C.

No. Copies	Agency
1	Director National Aeronautical Establishment Ottawa, Ontario Canada
1	University of Toronto Institute of Aerophysics ATTN: Library Toronto 5, Canada
1	Training Center for Experimental Aero-dynamics ATTN: Library Rhode-Saint-Genese (Belgique) 72, Chaussee de Waterloo, Brussels, Belgium
1	Library (Route to Dr. W.P. Jones) Aeronautical Research Council National Physical Laboratory Teddington, England
1	University of California Engineering Department ATTN: Libry.(Rte. to Drs. Leondes & Estrin) Los Angeles 24, California
1	California Institute of Technology ATTN: JPL Library 4800 Oak Grove Drive Pasadena 4, California
1	California Institute of Technology Guggenheim Aeronautical Laboratory ATTN: Library Pasadena 4, California
1	Harvard University Department of Engineering Sciences ATTN: Library Cambridge 38, Massachusetts
1	John Crerar Library 86 E. Randolph Street Chicago 1, Illinois
1	The Johns Hopkins University Department of Mechanics ATTN: Libry.(Rte. to Prof. F.H.Clauser) Baltimore 18, Maryland
1	University of Maryland Institute for Fluid Mechanics and Applied Mechanics College Park, Maryland
1	Massachusetts Institute of Technology ATTN: Aeronautics Library Cambridge 39, Massachusetts
1	Massachusetts Institute of Technology Naval Supersonic Laboratory Cambridge 39, Massachusetts

DISTRIBUTION LIST

No. Copies	Agency
1	Midwest Research Institute ATTN: Library 425 Volker Boulevard Kansas City 10, Missouri
1	Polytechnic Institute of Brooklyn ATTN: Library 333 Jay Street Brooklyn 1, New York
1	The James Forrestal Research Center Princeton University ATTN: Library Princeton, New Jersey
1	Rensselaer Polytechnic Institute ATTN: Libry. (Rte. to Dr. Fleishman) Troy, New York
1	Stanford Research Institute Documents Center (ATTN: Acquisitions) Menlo Park, California
1	Stanford University Department of Aeronautical Engineering ATTN: Library Stanford, California
1	New York University Institute of Mathematical Sciences ATTN: Library New York 3, New York
1	University of Wisconsin Department of Electrical Engineering ATTN: Library (Rte. to Prof. Higgins) Madison, Wisconsin
1	University of Michigan Instrumentation Engineering Programs Ann Arbor, Michigan
1	Columbia University Department of Electrical Engineering ATTN; Libry. (Rte. to Prof. G. Kranc) New York 27, New York
1	Yale University School of Engineering ATTN: Library New Haven 10, Connecticut
1	University of California Department of Electrical Engineering ATTN: Libry. (Rte. to Dr. L. Zadeh) Berkeley, California
1	New York University College of Engineering ATTN: Libry.(Rte. to Dean Ragazzini) New York, New York
1	Purdue University Department of Electrical Engineering ATTN: Libry. (Rte. to Prof. J.E. Gibson) Lafayette, Indiana

No. Copies	Agency
1	University of Illinois Department of Electrical Engineering ATTN: Libry.(Rte. to Dr. Van Vaulkenberg) Urbana, Illinois
1	Boeing Scientific Research Laboratories ATTN: Research Library P.O. Box 3981 Seattle 24, Washington
1	Convair Scientific Research Laboratory ATTN: Library P.O. Box 950 San Diego 12, California
1	Cornell Aeronautical Laboratories, Inc. ATTN: Library 1965 Sheridan Avenue Buffalo 23, New York
1	Grumman Aircraft Engineering Corporation ATTN: Library Bethpage, Long Island, New York
1	Hughes Aircraft Company Research and Development Laboratories ATTN: Library Culver City, California
1	Lockheed Missile Systems Division ATTN: Library Palo Alto, California
1	The Martin Company ATTN: Library Baltimore 3, Maryland
1	School of Aeronautical and Engineering Sciences Purdue University ATTN: Library (Route to Dr. Hu Lo) Lafayette, Indiana
2	Rand Corporation 1700 Main Street Santa Monica, California
1	Unified Science Associates, Inc. ATTN: S. Naiditch, President 926 S. Arroyo Parkway Pasadena, California
1	Westinghouse Electric Corporation Air Arm Division Friendship International Airport Baltimore, Maryland
1	Aeronca Manufacturing Corporation Aerospace Division P.O. Box 536 Baltimore 3, Maryland
1	Flight Sciences Laboratory ATTN: Library 1965 Sheridan Avenue Buffalo 23, New York

DISTRIBUTION LIST

No. Copies	Agency
1	General Electric Company Research Laboratory P.O. Box 1088 Schenectady 5, New York
1	MSVD Library General Electric Company Valley Forge Space Technology Center King of Prussia, Pennsylvania ATTN: L. Chasen, Manager - Library M/F Dr. F. W. Wendt